E-SOLID e-VARIETIES ARE NOT FINITELY BASED*

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ABSTRACT

It is proved that no non-orthodox e-variety of E-solid semigroups has a finite basis for its bi-identities.

1. Introduction and preliminaries

In order to study regular semigroups from the universal algebraist's point of view, Hall [7] invented the notion of an **existence variety** (briefly: **e-variety**) of regular semigroups: this is a class of regular semigroups closed under the formation of direct products, *regular* subsemigroups and morphic images. The same concept has been independently developed by Kaďourek and the second author [11] under the term **bivariety**, but for the subclass of orthodox semigroups

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only. For this latter class the theory turned out to be particularly nice: there is a suitable notion of an identity (called **bi-identity**) and a suitable concept of free object (called **bifree object**) such that all classical results from the theory of varieties of universal algebras have natural analogues in this context. For instance, a class is an e-variety if and only if it is definable by a set of bi-identities; there is a natural correspondence between the lattice of fully invariant congruences on the bifree orthodox semigroup (on an infinite set of generators) and the lattice of all e-varieties of orthodox semigroups, and so on.

The next fundamental result in this context was by Yeh [18] who discovered that bifree objects also exist outside the class of orthodox semigroups: namely, an e-variety $\mathcal{V}$ contains bifree objects on sets of arbitrary cardinality if and only if all members of $\mathcal{V}$ are either E-solid or locally inverse. Refining some ideas of Yeh, the first author designed in [1] a type of bi-identity which allowed to carry over to the locally inverse case all the universal algebraic results previously found for orthodox semigroups. (The key idea here is to expand the algebraic type; more precisely, to introduce a new binary operation which is naturally defined on each locally inverse semigroup.) In addition, concrete models of the bifree locally inverse semigroup have been constructed (see [1, 2]). A further important step was the construction of a model of the bifree E-solid semigroup by the second author in [15]. This result also suggested what an appropriate concept of bi-identity for E-solid semigroups should look like. Similarly to the locally inverse case, the algebraic type has to be expanded by a new — now unary — operation, which, however, is in contrast to the locally inverse case now no longer a total operation but a partial one. Nevertheless, all the classical universal algebraic results could be established for this case, too. This new concept and its consequences (including a Birkhoff type theorem, etc.) have been developed in [12].

An analogue to the Completeness Theorem of Equational Logic is also among the results which hold for e-varieties; this has been mentioned (but not formulated) for the orthodox and E-solid cases in [11, 12], and is implicit (but again not formulated) for the locally inverse case in [3]. Given such a theorem it is natural, for specific e-varieties $\mathcal{V}$, to search for a set Σ of bi-identities which hold in $\mathcal{V}$ and such that each bi-identity holding in $\mathcal{V}$ can be derived from the bi-identities in Σ. Such a basis Σ (consisting of four independent bi-identities) was found in [3] for the e-variety of all locally inverse semigroups. The purpose of the present paper is to prove that no non-orthodox E-solid e-variety has a finite basis for its bi-identities.

The paper is structured as follows. In the next section we recall from [12] all the relevant information that is required for the proper understanding of the particular type of bi-identity adequate for the syntactical description of E-solid semigroups. The Completeness Theorem of Equational Logic will be also formulated. Section 3 is devoted to the proof that there is no finite basis of bi-identities for E-solid semigroups. More generally, it will be shown that no non-orthodox e-variety of E-solid semigroups has a finite basis. In contrast, the e-variety of all orthodox semigroups does have a finite basis (even in the signature of E-solid semigroups); the latter will be demonstrated in Section 4. In the rest of the present section we shall give a few definitions and shall collect some results which will be needed in the sequel.

For background information in semigroup theory the reader is referred to Howie [8]; for basic concepts of universal algebra see Burris–Sankappanavar [4]. An introduction to e-varieties of regular semigroups in combination with a collection of results can be found in the survey articles by Jones [9] and Trotter [17].

As usual, for a semigroup S, $E(S)$ or E stands for the set of idempotents of S; for $x \in S$ let $V(x) = \{y \in S \mid x = xyx,\ y = yxy\}$ be the set of all inverses of x. The subsemigroup of S generated by $E(S)$ is the core of S, to be denoted by $C(S)$. It is well known that $C(S)$ is regular if S is regular. A subsemigroup T of a regular semigroup S is **selfconjugate** if for each $x \in S^1$, $x' \in V(x)$, the inclusion $xTx' \subseteq T$ holds. For any subsemigroup T of a regular semigroup S, we define its **conjugate** in S to be the subsemigroup of S generated by the set $\{sts' \mid t \in T,\ s \in S^1,\ s' \in V(s)\}$, and denote it by T_c. Putting $T_{c^n} = (T_{c^{n-1}})_c$ and letting $T_\infty = \bigcup_{n=1}^{\infty} T_{c^n}$ then for $T = C = C(S)$ we get the **selfconjugate core** $C_\infty(S)$ which is the least selfconjugate subsemigroup of S which contains the core; according to Trotter [16], $C_\infty(S)$ is a regular subsemigroup of S. A regular semigroup S is E-**solid** if for any idempotents $e, f, g \in E(S)$ such that $e\ \mathcal{R}\ f\ \mathcal{L}\ g$ there is an idempotent h such that $e\ \mathcal{L}\ h\ \mathcal{R}\ g$. Throughout we shall assume an E-solid semigroup automatically to be regular. It is well known that E-solid semigroups may be alternatively characterized as regular semigroups S with $C(S)$ being completely regular (see Hall [6]), or as regular semigroups S with $C_\infty(S)$ being completely regular (see Trotter [16]). The second characterization implies that each E-solid semigroup carries a partial unary operation $x \mapsto x^{-1}$ which is defined on $C_\infty(S)$ and which denotes the operation of taking the group inverse of x (within the maximal subgroup H_x of S containing x). For each non-empty set A denote by A^+ the absolutely free semigroup on A.

We turn to the definition of bifree objects. Let X be a non-empty set and

$X' = \{x' \mid x \in X\}$ be a disjoint copy of X, $x \mapsto x'$ being a bijection. A mapping θ: $X \cup X' \to S$ (a regular semigroup) is **matched** if $x'\theta \in V(x\theta)$ for each $x \in X$. Let $\mathcal{V}$ be a class of regular semigroups. A member F of $\mathcal{V}$ together with a matched mapping ι: $X \cup X' \to F$ is a **bifree object** on X in $\mathcal{V}$ if for each $S \in \mathcal{V}$ each matched mapping θ: $X \cup X' \to S$ admits a unique extension to a morphism $\bar{\theta}$: $F \to S$ (that is, there is a uniquely determined morphism $\bar{\theta}$: $F \to S$ satisfying $\iota\bar{\theta} = \theta$). Provided it exists, a bifree object is unique (up to the cardinality of X and up to isomorphic copies). This notion has been defined for the first time in [11] for classes of orthodox semigroups where it also has been shown that each e-variety of orthodox semigroups has a bifree object on each non-empty set X. Later it was proved by Yeh [18] that an e-variety $\mathcal{V}$ has a bifree object on each non-empty set if and only if $\mathcal{V}$ is either locally inverse or E-solid. Finally, some specific e-varieties will be denoted as follows:

$\mathcal{ES}$ – E-solid semigroups,
$\mathcal{O}$ – orthodox semigroups,
$\mathcal{CR}$ – completely regular semigroups,
$\mathcal{CS}$ – completely simple semigroups,
$\mathcal{G}$ – groups.

2. Equational logic for E-solid semigroups

We outline the theory of bi-identities for E-solid semigroups as developed by Kaďourek and the second author in [12]. Throughout, X denotes a fixed countably infinite set $x_1, x_2, \ldots$ of variables. Let $X' = \{x' \mid x \in X\}$ be a disjoint copy of X and put $\overline{X} = X \cup X'$. For $y = x' \in X'$ let $y' = x$; however, this must be understood just as a notational convenience: X and X' are two sorts of variables, and $'$ will never be used as an operational symbol. Further, let $U(\overline{X})$ be the free unary semigroup on $\overline{X}$; it is well known (see for example Clifford [5]) that $U(\overline{X})$ may be viewed as the smallest subsemigroup U of the absolutely free semigroup on the alphabet $\overline{X} \cup \{(,)^{-1}\}$ containing $\overline{X}$ and such that $u \in S$ implies $(u)^{-1} \in S$ (here (and $)^{-1}$ are two new symbols not contained in $\overline{X}$). Let $G(X)$ be the free group on X; then the mapping γ: $\overline{X} \to G(X)$, $x \mapsto x$, $x' \mapsto x^{-1}$ (where $x \in X$) extends uniquely to a morphism of unary semigroups $\overline{\gamma}$: $U(\overline{X}) \to G(X)$. Let $K(X) = 1\overline{\gamma}^{-1}$; notice that $K(X)$ consists precisely of all elements u of $U(\overline{X})$ whose "group reduced form" is the empty word (where the group reduced form is computed in an obvious way: apply $(uv)^{-1} \to (v)^{-1}(u)^{-1}$; $((u)^{-1})^{-1} \to u$; $(x)^{-1} \to x'$; $(x')^{-1} \to x$; $xx', x'x \to 1$ where $u, v \in U(\overline{X})$, $x \in X$ and 1 is the empty word). The crucial set of terms $T(X)$ is given by

Definition 2.1: Denote by $T(X)$ the smallest subsemigroup T of $U(\overline{X})$ containing $\overline{X}$ and such that for each $u \in T \cap K(X)$ also $(u)^{-1} \in T$. Further, put $R(X) = T(X) \cap K(X)$. Then $T(X)$ is a semigroup endowed with the partial unary operation $u \mapsto (u)^{-1}$ $(u \in R(X))$. The elements of $T(X)$ will be called **terms**.

Throughout the paper, $T(X)$ will be considered as a semigroup with a partial unary operation. So by a congruence on $T(X)$ we will mean an equivalence relation respecting both the multiplication and the partial unary operation. Notice that the set of terms $T(X)$ has been denoted by $F'^{\infty}(X)$ in [12]. It is clear what the subterms of a given term $u \in T(X)$ are. Formally the relation of being a subterm may be defined by induction: for all $u, v, w \in T(X)$, u is a subterm of u, u and v are subterms of uv, u is a subterm of $(u)^{-1}$, and if u is a subterm of v and v a subterm of w then u is also a subterm of w. Proofs for the results in this section can be found in [12].

RESULT 2.2: *Each element of $R(X)$ can be obtained from the terms xx' and $x'x$ $(x \in X)$ by a finite number of applications of the following operations:*
(1) *concatenation: form uv from the terms u and v,*
(2) *inversion: form $(u)^{-1}$ from u,*
(3) *conjugation: form $y'uy$ for $u \in R(X)$ and $y \in \overline{X}$.*

It is well known that the congruences and the morphisms of E-solid semigroups respect the partial unary operation of forming the group inverses of the elements in the selfconjugate core. So each E-solid semigroup can be considered as a semigroup endowed with this partial unary operation. By a morphism of $T(X)$ into an E-solid semigroup, we will mean a mapping respecting both the multiplication and the partial unary operation. The following is crucial for the definition of a bi-identity.

RESULT 2.3: *Let S be an E-solid semigroup and $\theta: \overline{X} \to S$ be a matched mapping. Then there is a unique morphism $\bar{\theta}: T(X) \to S$ which extends θ.*

We therefore define a **bi-identity** to be a string of the form $u \simeq v$ with $u, v \in T(X)$. A bi-identity **holds** in S or S **satisfies** the bi-identity $u \simeq v$ (denoted by $S \models u \simeq v$) if $u\bar{\theta} = v\bar{\theta}$ for the homomorphic extension $\bar{\theta}$ of each matched mapping $\theta: \overline{X} \to S$. A class $\mathcal{V}$ of E-solid semigroups satisfies $u \simeq v$ (denoted by $\mathcal{V} \models u \simeq v$) if $S \models u \simeq v$ for each $S \in \mathcal{V}$. If Σ is a set of bi-identities then we say that Σ **holds** in S or S **satisfies** Σ (denoted by $S \models \Sigma$) if $S \models u \simeq v$ for each $u \simeq v$ in Σ. For a class $\mathcal{V}$ of E-solid semigroups let $\Sigma(\mathcal{V})$ be the set of all bi-identities which hold in $\mathcal{V}$. Denote by $\rho(\mathcal{V})$ the binary relation on $T(X)$

corresponding to $\Sigma(\mathcal{V})$, that is, $(u, v) \in \rho(\mathcal{V})$ if and only if $u \simeq v \in \Sigma(\mathcal{V})$. So we have the following analogues to the classical results of universal algebra.

RESULT 2.4: *For each e-variety $\mathcal{V}$ of E-solid semigroups, $\rho(\mathcal{V})$ is a congruence on $T(X)$ (respecting the partial unary operation) and $T(X)/\rho(\mathcal{V})$ is a bifree object in $\mathcal{V}$ on X.*

For a set Σ of bi-identities let $[\Sigma]$ denote the class of all E-solid semigroups S satisfying Σ.

RESULT 2.5: *Let $\mathcal{V}$ be a class of E-solid semigroups. Then $\mathcal{V}$ is an e-variety if and only if $\mathcal{V} = [\Sigma]$ for some set Σ of bi-identities. If $\mathcal{V}$ is an e-variety then we have $\mathcal{V} = [\Sigma(\mathcal{V})]$.*

Next let $u, p, q \in T(X)$ and suppose that $pq \in R(X)$. Then $u(x \to p, x' \to q)$ denotes the term which is obtained from u by substituting p for all occurrences of x and q for all occurrences of x'. Notice that the condition $pq \in R(X)$ guarantees that $u(x \to p, x' \to q) \in T(X)$. A congruence ρ on $T(X)$ is said to be closed under **regular substitution** if $u \ \rho \ v$, $p \ \rho \ pqp$, $q \ \rho \ qpq$ with $pq \in R(X)$ imply $u(x \to p, x' \to q) \ \rho \ v(x \to p, x' \to q)$.

Definition 2.6: A congruence ρ on $T(X)$ is **bi-invariant** if $\rho(\mathcal{ES}) \subseteq \rho$ and ρ is closed under regular substitution.

RESULT 2.7: *A congruence ρ on $T(X)$ is of the form $\rho(\mathcal{V})$ for some e-variety $\mathcal{V}$ if and only if ρ is bi-invariant.*

The definition of a bi-invariant congruence in [12] is slightly different from that given here: there it is assumed that a bi-invariant congruence ρ is closed under simultaneous regular substitutions: $x_1 \to p_1, x_1' \to q_1, \ldots, x_n \to p_n, x_n' \to q_n$. However, due to the fact that we have infinitely many variables at hand it is clear that each such simultaneous substitution can be obtained as a finite sequence of regular substitutions as defined above. Consequently, the two definitions turn out to be equivalent.

The considerations so far suggest that we introduce the following concept of "biequational logic". In the sequel, for $p, q \in T(X)$ the string $p \in V(q)$ will simply be an abbreviation for the two bi-identities $p \simeq pqp$ and $q \simeq qpq$.

Definition 2.8: A set D of bi-identities is **deductively closed** if

1. (a) $u \simeq u \in D$ for all $u \in T(X)$,
   (b) $u \simeq v \in D$ implies $v \simeq u \in D$,
   (c) $u \simeq v, v \simeq w \in D$ imply $u \simeq w \in D$;

2. (a) $u \simeq v \in D$ implies $wu \simeq wv, uw \simeq vw \in D$ for all $w \in T(X)$,
   (b) $u \simeq v \in D$ and $u, v \in R(X)$ imply $(u)^{-1} \simeq (v)^{-1} \in D$;

3. (a) $x \simeq xx'x, x' \simeq x'xx' \in D$ for all $x \in X$,
   (b) $u \simeq v, p \in V(q) \in D, pq \in R(X)$ imply
   $u(x \to p, x' \to q) \simeq v(x \to p, x' \to q) \in D$.

For any set Σ of bi-identities, $D(\Sigma)$ denotes the smallest deductively closed set containing Σ and is called the **deductive closure** of Σ.

Definition 2.9: Let Σ be a set of bi-identities and $u \simeq v$ be a bi-identity. Then $u \simeq v$ is a **syntactical consequence** of Σ (denoted by $\Sigma \vdash u \simeq v$) if $u \simeq v$ can be **proved** from the bi-identities in Σ, that is, if there is a finite sequence $u_1 \simeq v_1, \ldots, u_n \simeq v_n$ of bi-identities such that $u_n \simeq v_n$ is just the bi-identity $u \simeq v$ and for each i, $u_i \simeq v_i$ is either of the form $p \simeq p$, $x \simeq xx'x$ ($p \in T(X)$, $x \in \overline{X}$), or $u_i \simeq v_i \in \Sigma$, or $u_i \simeq v_i$ can be obtained from some preceeding bi-identities by applying one of the rules given by (1b), (1c), (2a), (2b), (3b) in Definition 2.8.

By the use of standard arguments (see for example [4], Section II.14) one easily gets

RESULT 2.10: *For each set Σ of bi-identities,*

$$D(\Sigma) = \{u \simeq v \mid \Sigma \vdash u \simeq v\}.$$

As pointed out in [12] we have an analogue to Birkhoff's Completeness Theorem of Equational Logic. In order to formulate it in a compact form we still need another definition.

Definition 2.11: Let Σ be a set of bi-identities; a bi-identity $u \simeq v$ is a **semantical consequence** of Σ (denoted by $\Sigma \models u \simeq v$) if for each E-solid semigroup S such that $S \models \Sigma$ we also have $S \models u \simeq v$.

The mentioned completeness theorem now may be formulated as follows (its proof is analogous to the proof of the classical one for varieties of universal algebras and follows from the results of [12]).

THEOREM 2.12: *For any set Σ of bi-identities and for any bi-identity $u \simeq v$ we have*

$$\Sigma \models u \simeq v \quad \textit{if and only if } \Sigma \cup \Sigma(\mathcal{ES}) \vdash u \simeq v.$$

In this theorem the set $\Sigma(\mathcal{ES})$ of all bi-identities which hold in $\mathcal{ES}$ may be replaced by any subset $\Sigma_1 \subseteq \Sigma(\mathcal{ES})$ with $D(\Sigma_1) = \Sigma(\mathcal{ES})$. For instance, as pointed out in [12], one may choose

$$\Sigma_1 = \{(u)^{-1}u \simeq u(u)^{-1}, u(u)^{-1}u \simeq u, ((u)^{-1})^{-1} \simeq u \mid u \in R(X)\}.$$

But this set of bi-identities is obviously infinite. We are naturally led to the question if we can replace $\Sigma(\mathcal{ES})$ by a finite subset Σ_1. The purpose of this paper is to show that such a finite set does not exist. Call a set Σ of bi-identities a **basis** for the bi-identities of an e-variety $\mathcal{V}$ if $D(\Sigma) = \Sigma(\mathcal{V})$. We will prove in the next section that no non-orthodox E-solid e-variety $\mathcal{V}$ has a finite basis for its bi-identities. In contrast, $\mathcal{O}$ does have a finite basis for its bi-identities, namely $\Sigma = \{(xx'yy')^2 \simeq xx'yy', (xx'yy')^{-1} \simeq xx'yy'\}$. The latter will be demonstrated in Section 4.

3. Non-orthodox E-solid e-varieties are not finitely based

In order to prove the statement in the title we need some prerequisites. For a prime number q let $\mathcal{CS}(\mathcal{A}_q)$ denote the (e-)variety of all completely simple semigroups all of whose subgroups are abelian of exponent q. Let $\mathcal{V}$ be any E-solid e-variety which is not orthodox. Then $\mathcal{V}$ contains a non-orthodox completely simple semigroup S, and therefore contains a completely simple semigroup $\mathcal{M}\left(\mathbf{2}, \langle a \rangle, \mathbf{2}, \begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix}\right)$ where $\mathbf{2}$ is any 2-element set and $\langle a \rangle$ is a non-trivial cyclic group generated by a. In particular, $\mathcal{V}$ contains some non-orthodox e-variety of completely simple semigroups all of whose subgroups are abelian. From the description of the lattice of (e-)varieties of completely simple semigroups with abelian subgroups given by Petrich and Reilly in [14], Theorem VIII.9.3, it follows that each non-orthodox e-variety of completely simple semigroups (all of whose subgroups are abelian) contains an e-variety $\mathcal{CS}(\mathcal{A}_q)$ for some prime number q. Therefore, from the remark above one can see that each non-orthodox e-variety of E-solid semigroups also contains some $\mathcal{CS}(\mathcal{A}_q)$. We can formulate

RESULT 3.1: *Each non-orthodox e-variety of E-solid semigroups contains the e-variety $\mathcal{CS}(\mathcal{A}_q)$ for some prime q.*

From Corollary 2.5 in [10] it follows that a model of the bifree semigroup on $X_0 = X \cup \{x_0\}$ (with $x_0 \notin \overline{X}$) in $\mathcal{CS}(\mathcal{A}_q)$ can be constructed as follows. Let $Z = \{p_{xy} \mid x, y \in \overline{X_0} \setminus \{x_0\}\}$ be a set of new symbols (that is, $Z \cap \overline{X_0} = \emptyset$) indexed by all pairs (x, y) of elements of $\overline{X_0} \setminus \{x_0\}$ and let A be the free abelian group of exponent q generated by $Z \cup X_0$. Next let $p_{x_0 x} = p_{x x_0} = 1$, the identity of A, for each $x \in X_0$ and let P be the $\overline{X_0} \times \overline{X_0}$-matrix $P = (p_{xy})$. Denote by F the Rees matrix semigroup $F = \mathcal{M}(\overline{X_0}, A, \overline{X_0}; P)$. Then F, together with the (matched) mapping $\iota\colon \overline{X_0} \to F$, $x \mapsto (x,x,x)$, $x' \mapsto (x', p_{xx'}^{-1} x^{-1} p_{x'x}^{-1}, x')$ ($x \in X_0$) is a bifree object on X_0 in $\mathcal{CS}(\mathcal{A}_q)$. Notice that F depends on q, and in the following we shall assume to have the prime q fixed. Denote by $A(Z)$ the subgroup of A generated by Z. Finally, recall that each endomorphism θ of F can be represented by a certain triple (ϕ, ω, ψ) where ϕ and ψ are mappings $\overline{X_0} \to \overline{X_0}$ and ω is an endomorphism of A such that (ϕ, ω, ψ) satisfies certain compatibility conditions (for more precise information on endomorphisms of completely simple semigroups the reader is referred to results III.3.10 and III.3.11 in [14] or to Lemma 2.1 in [13]). All we have to take into account in the following is that each endomorphism of a completely simple semigroup maps $\mathcal{L}$-classes into $\mathcal{L}$-classes and $\mathcal{R}$-classes into $\mathcal{R}$-classes, and in the above-mentioned triple notation (ϕ, ω, ψ) of an endomorphism θ, for each $(x,g,y) \in F$, we have $(x,g,y)\theta = (x\phi, h, y\psi)$ for some $h \in A$. Denote by $\Phi\colon T(X) \to F$ the unique morphism determined by the matched mapping $x \mapsto (x,x,x)$, $x' \mapsto (x', p_{xx'}^{-1} x^{-1} p_{x'x}^{-1}, x')$. Notice that $x_0 \notin X$ and so for all matrix elements involved $p_{xy} \neq 1$ holds. The following technical lemma is crucial in that it gives us a means to measure — in some sense — the complexity of a term $u \in T(X)$.

LEMMA 3.2:

(i) *For each $u \in R(X)$, $u\Phi = (x,a,y)$ for some $x,y \in \overline{X}$ and some $a \in A(Z)$.*

(ii) *Let θ be an endomorphism of F represented by the triple (ϕ, ω, ψ) and let π be the endomorphism of $A(Z)$ determined by $p_{xy} \mapsto p_{x\psi, y\phi}$. Then for each $u \in R(X)$, if $u\Phi = (x,a,y)$ then $(u\Phi)\theta = (x\phi, a\pi, y\psi)$.*

Proof: The argument is by induction on the complexity of the term u (according to Result 2.2), and (i) and (ii) are shown simultaneously. Obviously, $(xx')\Phi = (x, p_{x'x}^{-1}, x')$ and $(x'x)\Phi = (x', p_{xx'}^{-1}, x)$ where $x \in X$ and $p_{x'x}^{-1}, p_{xx'}^{-1} \in A(Z)$. Moreover, $((xx')\Phi)\theta$ and $((x'x)\Phi)\theta$ are the idempotents of the $\mathcal{H}$-classes corresponding to $(x\phi, x'\psi)$ and $(x'\phi, x\psi)$, respectively. Therefore $((xx')\Phi)\theta = (x\phi, p_{x'\psi, x\phi}^{-1}, x'\psi) = (x\phi, p_{x'x}^{-1}\pi, x'\psi)$ and $((x'x)\Phi)\theta = (x'\phi, p_{x\psi, x'\phi}^{-1}, x\psi) = (x'\phi, p_{xx'}^{-1}\pi, x\psi)$.

Now suppose that $u, v \in R(X)$, $u\Phi = (x,a,y)$, $v\Phi = (w,b,z)$ with $x,y,w,z \in \overline{X}$ and $a, b \in A(Z)$. Furthermore, assume that $(u\Phi)\theta = (x\phi, a\pi, y\psi)$ and $(v\Phi)\theta = (w\phi, b\pi, z\psi)$. Then $(uv)\Phi = (u\Phi)(v\Phi) = (x, ap_{yw}b, z)$ where $x, z \in \overline{X}$ and $ap_{yw}b \in A(Z)$ and $((uv)\Phi)\theta = (u\Phi)\theta(v\Phi)\theta = (x\phi, a\pi, y\psi)(w\phi, b\pi, z\psi) = (x\phi, a\pi \cdot p_{y\psi, w\phi} \cdot b\pi, z\psi) = (x\phi, (ap_{yw}b)\pi, z\psi)$. One can also see directly that $(u)^{-1}\Phi = (u\Phi)^{-1} = (x, p_{yx}^{-1} a^{-1} p_{yx}^{-1}, y)$ and, since θ commutes with taking group inverses, we obtain that

$$\begin{aligned}((u)^{-1}\Phi)\theta &= (u\Phi)^{-1}\theta \\ &= ((u\Phi)\theta)^{-1} \\ &= (x\phi, a\pi, y\psi)^{-1} \\ &= (x\phi, p_{y\psi,x\phi}^{-1}(a\pi)^{-1}p_{y\psi,x\phi}^{-1}, y\psi) \\ &= (x\phi, (p_{yx}^{-1}a^{-1}p_{yx}^{-1})\pi, y\psi).\end{aligned}$$

Thus properties (i) and (ii) in the statement of the lemma are preserved by operations (1) and (2) of Result 2.2.

Consider now operation (3). We can easily check that

$$\begin{aligned}(zuz')\Phi &= z\Phi \cdot u\Phi \cdot z'\Phi \\ &= (z,z,z)(x,a,y)(z', p_{zz'}^{-1}z^{-1}p_{z'z}^{-1}, z') \\ &= (z, zp_{zx}ap_{yz'}p_{zz'}^{-1}z^{-1}p_{z'z}^{-1}, z') \\ &= (z, p_{zx}ap_{yz'}p_{zz'}^{-1}p_{z'z}^{-1}, z')\end{aligned}$$

by commutativity of A, and here we have $z, z' \in \overline{X}$ and $p_{zx}ap_{yz'}p_{zz'}^{-1}p_{z'z}^{-1} \in A(Z)$. Now if $(z\Phi)\theta = (z,z,z)\theta = (z\phi, c, z\psi)$ for some $c \in A$ then $(z'\Phi)\theta$ is the inverse of $(z\Phi)\theta$ in the $\mathcal{H}$-class corresponding to $(z'\phi, z'\psi)$, that is, $(z'\Phi)\theta = (z'\phi, p_{z\psi,z'\phi}^{-1}c^{-1}p_{z'\psi,z\phi}^{-1}, z'\psi)$. Applying θ we obtain that

$$\begin{aligned}((zuz')\Phi)\theta &= (z\Phi)\theta \cdot (u\Phi)\theta \cdot (z'\Phi)\theta \\ &= (z\phi, c, z\psi)(x\phi, a\pi, y\psi)(z'\phi, p_{z\psi,z'\phi}^{-1}c^{-1}p_{z'\psi,z\phi}^{-1}, z'\psi) \\ &= (z\phi, cp_{z\psi,x\phi} \cdot a\pi \cdot p_{y\psi,z'\phi}p_{z\psi,z'\phi}^{-1}c^{-1}p_{z'\psi,z\phi}^{-1}, z'\psi) \\ &= (z\phi, (p_{zx}ap_{yz'}p_{zz'}^{-1}p_{z'z}^{-1})\pi, z'\psi)\end{aligned}$$

by commutivity of A. Thus we see that properties (i) and (ii) in the statement of the lemma are preserved also by operation (3) of Result 2.2. ∎

Each $a \in A$ can be uniquely written, up to the order, as $a = z_1^{\alpha_1} \cdots z_s^{\alpha_s}$ where z_i are pairwise distinct elements from $X_0 \cup Z$ and $0 < \alpha_1, \ldots, \alpha_s < q$. In particular, in case $a = 1$ we have $s = 0$. This enables us to introduce the next auxiliary concept.

Definition 3.3: For $f = (x,a,y) \in F$ put $\#f = s$ if $a = z_1^{\alpha_1} \cdots z_s^{\alpha_s}$ as in the representation above, and for $u \in T(X)$ let $\#u = \#(u\Phi)$.

Notice that item (ii) of Lemma 3.2 implies that for $u \in R(X)$ and for each endomorphism θ of F, we have $\#(u\Phi\theta) \leq \#(u\Phi)$. This allows us to obtain another lemma for later use.

LEMMA 3.4: *Let $u \in R(X)$ and $p, q \in T(X)$ be such that $pq \in R(X)$ and $F \models p \in V(q)$. Then $\#(u(x \to p, x' \to q)) \leq \#u$.*

Proof: Let θ be the endomorphism of F determined by $x\Phi \mapsto p\Phi$, $x'\Phi \mapsto q\Phi$ and $y\Phi \mapsto y\Phi$ for all $y \in \overline{X}$ with $y \neq x, x'$. There is such an endomorphism since F is a bifree object and $p\Phi$ and $q\Phi$ are mutually inverse elements of F (and so the above mapping is matched). Then $(u\Phi)\theta = (u(x \to p, x' \to q))\Phi$ and thus, as mentioned above, $\#(u(x \to p, x' \to q)) = \#(u\Phi\theta) \leq \#(u\Phi) = \#u$. ∎

In the following it will be convenient to use the concept of a **multiset** (as it is used in combinatorics) where a multiset is merely a set in which elements may occur several times and each element "counts" as often as it occurs, that is, a multiset is an unordered sequence. (Formally one can define a finite multiset as an element of a free commutative monoid.) So if we speak about the multiset of subterms of a term w then this is just the set of subterms but each subterm is counted as often as it occurs. For example, the multiset of subterms of the term xxx is $[x, x, x, xx, xx, xxx]$ which consists of six elements. We indicate multisets by brackets $[\ldots]$ rather than by braces $\{\ldots\}$.

Definition 3.5: Let $u \in T(X)$ and let M be a positive integer; let $S_M(u)$ be the multiset of subterms v of u such that $(v)^{-1}$ is also a subterm of u for that particular occurrence of v and $\#v > M$.

If u is a term such that for each subterm $(v)^{-1}$ of u, $\#v \leq M$ then $S_M(u) = \emptyset$, the empty (multi)set.

In the following we shall prove that no non-orthodox E-solid e-variety $\mathcal{V}$ has a finite basis for its bi-identities. In other words, given any non-orthodox E-solid e-variety $\mathcal{V}$, there is no finite set Σ of bi-identities such that $D(\Sigma) = \Sigma(\mathcal{V})$. The proof we shall present is as follows: if Σ is a set of bi-identities such that for some positive integer M, $S_M(u) = \emptyset$ for each term u occurring on one of the sides in a bi-identity of Σ then for each derived bi-identity $u \simeq v \in D(\Sigma)$ there is a bijective correspondence $w \mapsto \tilde{w}$ between $S_M(u)$ and $S_M(v)$. On the other hand, we shall see that for each positive integer M, there is a bi-identity $u \simeq v \in \Sigma(\mathcal{ES})$ such that $S_M(u) = \emptyset$ but $S_M(v) \neq \emptyset$.

LEMMA 3.6: *Let q be a prime, M be a positive integer, $\Sigma = \{u_i \simeq v_i \mid i \in I\}$ be a set of bi-identities such that $\Sigma \subseteq \Sigma(\mathcal{CS}(\mathcal{A}_q))$. Suppose that $S_M(u_i) = \emptyset = S_M(v_i)$ for all i. Then for each bi-identity $u \simeq v \in D(\Sigma)$ there is a bijective correspondence $w \mapsto \tilde{w}$ between $S_M(u)$ and $S_M(v)$ such that $w \simeq \tilde{w} \in D(\Sigma)$ for each corresponding pair $w \in S_M(u)$ and $\tilde{w} \in S_M(v)$.*

Notation 3.7: For convenience, we write $S_M(u) \leftrightarrow S_M(v)$ if $u \simeq v$ enjoys the property stated in the lemma.

Proof: For each bi-identity $u_i \simeq v_i \in \Sigma$ we trivially have that $S_M(u_i) \leftrightarrow S_M(v_i)$ since $S_M(u_i) = \emptyset = S_M(v_i)$. Further, all bi-identites $p \simeq p$ with $p \in T(X)$ and $x \simeq xx'x$ with $x \in \overline{X}$ have this property. Equally trivially, $S_M(u) \leftrightarrow S_M(v)$ implies $S_M(v) \leftrightarrow S_M(u)$ and likewise $S_M(u) \leftrightarrow S_M(v)$ and $S_M(v) \leftrightarrow S_M(w)$ imply $S_M(u) \leftrightarrow S_M(w)$. This means that the application of the rules (1b), (1c) of Definition 2.8 produces only bi-identities having the required property. It is further obvious that $S_M(u) \leftrightarrow S_M(v)$ implies $S_M(wu) \leftrightarrow S_M(wv)$ and $S_M(uw) \leftrightarrow S_M(vw)$ for each $w \in T(X)$. For, if $(t)^{-1}$ is a subterm of wu, say, it must be a subterm of w or u. Thus the application of rule (2a) still produces bi-identities having the required property.

Now let $u \simeq v \in D(\Sigma)$ for $u, v \in R(X)$ and suppose that $S_M(u) \leftrightarrow S_M(v)$. Application of rule (2b) gives $(u)^{-1} \cong (v)^{-1} \in D(\Sigma)$. Further, $S_M((u)^{-1}) = S_M(u)$ or $S_M((u)^{-1}) = S_M(u) \cup [u]$ depending on whether $\#u \leq M$ or $\#u > M$ and the analogous statement holds for $S_M((v)^{-1})$. But $u \simeq v \in D(\Sigma)$ implies $F \models u \simeq v$, so that $u\Phi = v\Phi$ and thus $\#u = \#v$. Therefore $S_M((u)^{-1}) = S_M(u) \cup [u]$ if and only if $S_M((v)^{-1}) = S_M(v) \cup [v]$ whence it follows that the bijective correspondence between $S_M(u)$ and $S_M(v)$ extends to such a correspondence between $S_M((u)^{-1})$ and $S_M((v)^{-1})$ by putting $u \mapsto \tilde{u} = v$ if required. So rule (2b) still produces only bi-identities enjoying the property we need.

Next observe that for $p, q \in T(X)$ with $pq \in R(X)$, both relations $S_M(p) \leftrightarrow S_M(pqp)$ and $S_M(q) \leftrightarrow S_M(qpq)$ are only possible if $S_M(p) = \emptyset = S_M(q)$. Let us consider rule (3b) of Definition 2.8 (regular substitution). Let $u, v, p, q \in T(X)$ be such that $pq \in R(X)$ and $u \simeq v$, $p \simeq pqp$, $q \simeq qpq$ are in $D(\Sigma)$, and suppose that $S_M(u) \leftrightarrow S_M(v)$, $S_M(p) \leftrightarrow S_M(pqp)$, $S_M(q) \leftrightarrow S_M(qpq)$. As already mentioned, the latter two relations imply $S_M(p) = \emptyset = S_M(q)$. Consider now the terms $u(x \to p, x' \to q)$ and $v(x \to p, x' \to q)$. Let t be a term such that $(t)^{-1}$ is a subterm of $u(x \to p, x' \to q)$. Then either $(t)^{-1}$ occurs as a subterm of p or q, or $t = w(x \to p, x' \to q)$ for some subterm w of u such that $(w)^{-1}$ is a subterm of u. In the first case, as we have seen above, we have $\#t \leq M$. In the second one, if $w \notin S_M(u)$ then by Lemma 3.4, we have $\#t \leq \#w$ and therefore $t \notin S_M(u(x \to p, x' \to q))$. So we have that

$$S_M(u(x \to p, x' \to q)) = [t(x \to p, x' \to q) \mid t \in S_M(u), \#t(x \to p, x' \to q) > M]$$

and an analogous statement holds for $S_M(v(x \to p, x' \to q))$. For $t \in S_M(u)$ denote by $\tilde{t}$ the unique element of $S_M(v)$ corresponding to t according to the correspondence $S_M(u) \leftrightarrow S_M(v)$. Then $t \simeq \tilde{t} \in D(\Sigma)$ implies

$$t(x \to p, x' \to q) \simeq \tilde{t}(x \to p, x' \to q) \in D(\Sigma)$$

whence

$$F \models t(x \to p, x' \to q) \simeq \tilde{t}(x \to p, x' \to q)$$

and so

$$\#t(x \to p, x' \to q) = \#\tilde{t}(x \to p, x' \to q)$$

follows. Thus

$$t(x \to p, x' \to q) \in S_M(u(x \to p, x' \to q))$$

if and only if

$$\tilde{t}(x \to p, x' \to q) \in S_M(v(x \to p, x' \to q)).$$

In other words, the bijective correspondence $\leftrightarrow$ between $S_M(u)$ and $S_M(v)$ carries over to the required correspondence between $S_M(u(x \to p, x' \to q))$ and $S_M(v(x \to p, x' \to q))$. ∎

Remark 3.8: In the proof of Lemma 3.6 one can see why we used multisets rather than sets for the definition of $S_M(u)$. If we had defined $S_M(u)$ to be the underlying *set* of subterms then the steps $S_M(u) \leftrightarrow S_M(v) \Rightarrow S_M(wu) \leftrightarrow S_M(wv)$, $S_M(uw) \leftrightarrow S_M(vw)$ would no longer be true in general. Likewise, the implication $S_M(pqp) \leftrightarrow S_M(p)$, $S_M(qpq) \leftrightarrow S_M(q) \Rightarrow S_M(p) = \emptyset = S_M(q)$ would also fail.

The main result is now an easy exercise.

THEOREM 3.9: *No non-orthodox E-solid e-variety has a finite basis for its bi-identities.*

Proof: Let $\mathcal{V}$ be a non-orthodox E-solid e-variety. From Result 3.1 there is a prime q such that $\mathcal{CS}(\mathcal{A}_q) \subseteq \mathcal{V}$ and therefore $\Sigma(\mathcal{V}) \subseteq \Sigma(\mathcal{CS}(\mathcal{A}_q))$. Let $\Sigma = \{u_1 \simeq v_1, \ldots, u_n \simeq v_n\}$ be a finite set of bi-identities such that $\Sigma \subseteq \Sigma(\mathcal{V})$. Define # with respect to the prime q and choose a positive integer $M \geq 2$ such that $S_M(u_i) = \emptyset = S_M(v_i)$ for each $i = 1, \ldots, n$. By Lemma 3.6, we have $S_M(u) \leftrightarrow S_M(v)$ for each $u \simeq v \in D(\Sigma)$. The bi-identity

$$x_1x_1' \cdots x_Mx_M' \simeq ((x_1x_1' \cdots x_Mx_M')^{-1})^{-1}$$

holds in $\mathcal{V}$. A straightforward calculation gives $\#(x_1x_1' \cdots x_Mx_M') = 2M - 1$ and $\#((x_1x_1' \cdots x_Mx_M')^{-1}) = 2M$ if $q > 2$ and $\#((x_1x_1' \cdots x_Mx_M')^{-1}) = 2M - 1$ if $q = 2$. In any case,

$$S_M(((x_1x_1' \cdots x_Mx_M')^{-1})^{-1}) = [x_1x_1' \cdots x_Mx_M', (x_1x_1' \cdots x_Mx_M')^{-1}]$$

whereas $S_M(x_1x_1' \cdots x_Mx_M') = \emptyset$. Consequently,

$$S_M(x_1x_1' \cdots x_Mx_M') \not\leftrightarrow S_M(((x_1x_1' \cdots x_Mx_M')^{-1})^{-1})$$

and therefore

$$\Sigma \not\vdash x_1x_1' \cdots x_Mx_M' \simeq ((x_1x_1' \cdots x_Mx_M')^{-1})^{-1}. \quad \blacksquare$$

COROLLARY 3.10: *$\mathcal{ES}$ has no finite basis for its bi-identities.*

In the classical context of varieties of universal algebras the property of being not finitely based is — in a sense — an "inconvenient" property. Indeed, it follows from the Completeness Theorem of First Order Logic that a variety is finitely based if and only if it is finitely axiomatizable as a class of first order structures. The same holds for e-varieties of orthodox semigroups in the signature of [11] and for e-varieties of locally inverse semigroups in the signature of [3]. In these cases we also have that an e-variety is finitely based if and only if it is a strictly elementary class. In the present context of e-varieties of E-solid semigroups the property of "overall" non-finite basedness does not reflect such inconveniences of the underlying classes — $\mathcal{ES}$ is strictly elementary and likewise are, for example, $\mathcal{CR}$ and $\mathcal{CS}$ which are finitely based as varieties of unary semigroups — but rather seems to reflect a deficiency of the signature itself coming from the fact that the unary operation $^{-1}$ is not total.

From this point of view it might be even surprising that there do exist non-trivial e-varieties which are finitely based in this signature, as will be shown in the next section.

4. Orthodox semigroups are finitely based

THEOREM 4.1: *The bi-identities $(xx'yy')^2 \simeq xx'yy'$ and $(xx'yy')^{-1} \simeq xx'yy'$ form a basis for the bi-identities of all orthodox semigroups.*

Proof: Let $\Sigma = \{(xx'yy')^2 \simeq xx'yy', (xx'yy')^{-1} \simeq xx'yy'\}$ and denote by ρ the congruence on $T(X)$ given by $u \mathrel{\rho} v$ if and only if $\Sigma \vdash u \simeq v$. We have to show that $\rho = \rho(\mathcal{O})$ where $\rho(\mathcal{O}) \subseteq \rho$ is the non-trivial inclusion. In order to prove this inclusion it is sufficient to prove that (i) $x'\rho \in V(x\rho)$ for all $x \in X$, (ii) $T(X)/\rho$ is orthodox and (iii) $u \mathrel{\rho} (u)^{-1}$ for each $u \in R(X)$. For in this case the mapping $\overline{X} \to T(X)/\rho$ is matched and extends to a morphism $T(X)/\rho(\mathcal{O}) \to T(X)/\rho$. Condition (i) trivially holds. The rest follows from a series of lemmas.

LEMMA 4.2: *Let* $p_1, \ldots, p_n, q_1, \ldots, q_n \in T(X)$ *be such that* $p_i q_i \in R(X)$ *and* $\Sigma \vdash q_i \in V(p_i)$ *for all* i. *Then* $\Sigma \vdash q_n \cdots q_1 \in V(p_1 \cdots p_n)$.

Proof: The claim is trivial for $n = 1$ so consider the case $n = 2$. The substitution $x \to q_1, x' \to p_1,\ y \to p_2, y' \to q_2$ in the first bi-identity gives $(q_1p_1p_2q_2)^2 \simeq q_1p_1p_2q_2$. So we get

$$\begin{aligned} p_1 \cdot p_2q_2q_1p_1 \cdot p_2 &\simeq p_1q_1p_1 \cdot p_2q_2q_1p_1 \cdot p_2q_2p_2 \\ &= p_1 \cdot q_1p_1p_2q_2 \cdot q_1p_1p_2q_2 \cdot p_2 \\ &\simeq p_1 \cdot q_1p_1p_2q_2 \cdot p_2 \\ &= p_1q_1p_1 \cdot p_2q_2p_2 \simeq p_1p_2 \end{aligned}$$

and by analogy $q_2q_1p_1p_2q_2q_1 \simeq q_2q_1$. Now we proceed by induction. Suppose $\Sigma \vdash q_n \cdots q_1 \in V(p_1 \cdots p_n)$ has already been proved. Then using the case $n = 2$ we get $q_{n+1} \cdot q_n \cdots q_1 \in V(p_1 \cdots p_n \cdot p_{n+1})$. ∎

COROLLARY 4.3: $\Sigma \vdash (xyy'x')^2 \simeq xyy'x' \simeq (xyy'x')^{-1}$.

Proof: The first bi-identity follows from $y'x' \in V(xy)$ proven in Lemma 4.2. For the second one substitute $y \to xy, y' \to y'x'$ in $(xx'yy')^{-1} \simeq xx'yy'$. ∎

LEMMA 4.4: *For each* $w \in \overline{X}^+$, $w\rho$ *is regular in* $T(X)/\rho$. *For each* $w \in \overline{X}^+ \cap R(X)$, $w^2 \;\rho\; w \;\rho\; (w)^{-1}$.

Proof: The first assertion is immediate from Lemma 4.2. Suppose that $w \in \overline{X}^+ \cap R(X)$. Then either $w = xw_0x'$ for some $w_0 \in \overline{X}^+ \cap R(X)$ and $x \in \overline{X}$ or $w = w_0w_1$ for some $w_0, w_1 \in \overline{X}^+ \cap R(X)$. The argument is by induction on the length of w. In case $w = xx'$ for some $x \in \overline{X}$, the assertion is trivial. Suppose that the claim is true for all words having length smaller than the length of w. In case $w = xw_0x'$ we have $w_0^2 \;\rho\; w_0$ whence $w_0\rho \in V(w_0\rho)$, which by Lemma 4.2 implies $(w_0x')\rho \in V((xw_0)\rho)$ so that $(xw_0w_0x')^2 \;\rho\; xw_0w_0x'$. Using once more that $w_0^2 \;\rho\; w_0$ we get $(xw_0x')^2 \;\rho\; xw_0x'$. Further, $xw_0w_0x' \;\rho\; (xw_0w_0x')^{-1}$ follows from $w_0\rho \in V(w_0\rho)$ and Corollary 4.3. Replacement of w_0^2 by w_0 implies that $xw_0x' \;\rho\; (xw_0x')^{-1}$. In case $w = w_0w_1$ we have $w_0^2 \;\rho\; w_0$ and $w_1^2 \;\rho\; w_1$ by the induction hypothesis, so that $w_0\rho \in V(w_0\rho)$ and $w_1\rho \in V(w_1\rho)$. Appropriate substitution in Σ yields $(w_0^2w_1^2)^2 \;\rho\; w_0^2w_1^2 \;\rho\; (w_0^2w_1^2)^{-1}$. Again using $w_0^2 \;\rho\; w_0$ and $w_1^2 \;\rho\; w_1$ we get $(w_0w_1)^2 \;\rho\; w_0w_1 \;\rho\; (w_0w_1)^{-1}$, as required. ∎

LEMMA 4.5: *For each* $w \in T(X)$ *there exists* $w_1 \in \overline{X}^+$ *such that* $w \;\rho\; w_1$.

Proof: The assertion is trivial for $w = x \in \overline{X}$. Let $u, v \in T(X)$ be such that for certain $u_1, v_1 \in \overline{X}^+$, $u \;\rho\; u_1$ and $v \;\rho\; v_1$. Then $uv \;\rho\; u_1v_1$ and $u_1v_1 \in \overline{X}^+$. Now let $u \in R(X)$ and suppose that $u \,\rho\, u_1$ for $u_1 \in \overline{X}^+$. Then $u_1 \in R(X)$ since $\rho \subseteq \rho(\mathcal{G})$. So $(u_1)^{-1}$ is meaningful and by Lemma 4.4 we have that $(u_1)^{-1} \;\rho\, u_1$. From $(u)^{-1} \;\rho\; (u_1)^{-1}$ we obtain $(u)^{-1} \;\rho\, u_1$, as required. ∎

So we return to the proof of Theorem 4.1 and show items (ii) and (iii). Lemma 4.4 and Lemma 4.5 guarantee that $T(X)/\rho$ is regular. Let $e, f \in E(T(X)/\rho)$. Since $\rho \subseteq \rho(\mathcal{G})$ we have that $e = u\rho$, $f = v\rho$ for some $u, v \in R(X)$ and therefore $uv \in R(X)$. From Lemma 4.5, $uv \;\rho\; w$ for some $w \in \overline{X}^+$, and, again applying $\rho \subseteq \rho(\mathcal{G})$, we obtain $w \in R(X)$. From Lemma 4.4 we obtain $w \;\rho\; w^2$, that is, $(ef)^2 = ef$ so that $T(X)/\rho$ is orthodox. Finally, item (iii) follows from Lemma 4.4 in combination with Lemma 4.5. ∎

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